Improving multifrontal methods by means of low-rank approximation techniques

Joint work with Patrick Amestoy , Cleve Ashcraft, Olivier Boiteau, Alfredo Buttari and Jean-Yves L'Excellent, PhD started on October 1st, 2010 and financed by EDF.

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SIAM LA12, Valencia, Spain, June 22, 2012

Introduction

Improving **MULTIFRONTAL METHODS** by means of
LOW-RANK APPROXIMATION TECHNIQUES

Multifrontal solver

- direct solver for large linear systems
- well known and studied

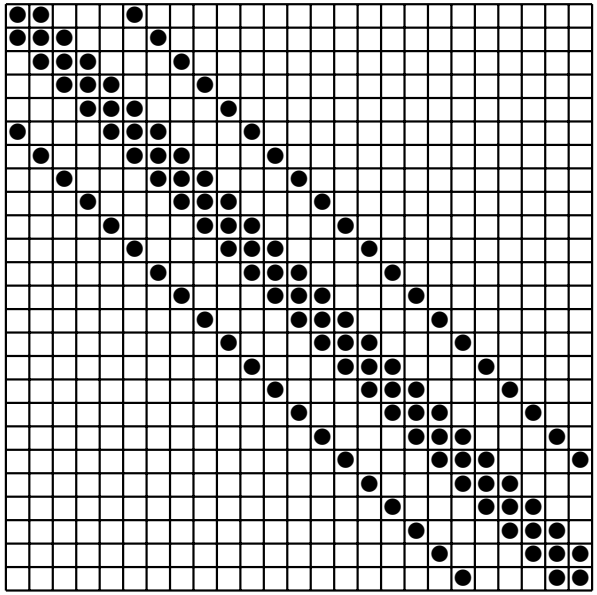
Low-rank approximations

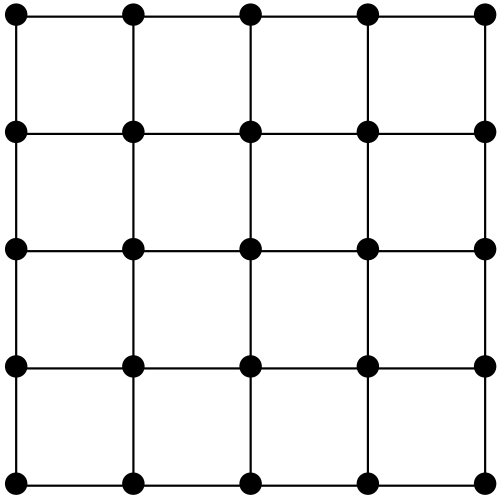
- compression and flop reduction
- accuracy controlled by a numerical parameter

⇒ Combine these two notions to improve multifrontal solvers (in the context of `MUMPS`)

The multifrontal method

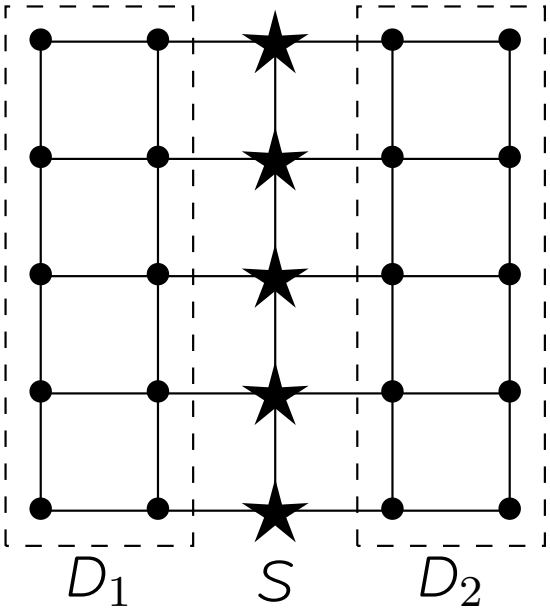
From matrices to graphs





- 1 variable of the matrix A = 1 node in the graph $\mathcal{G}$
- $A(i,j) \neq 0 \Leftrightarrow$ an edge between nodes i and j exists

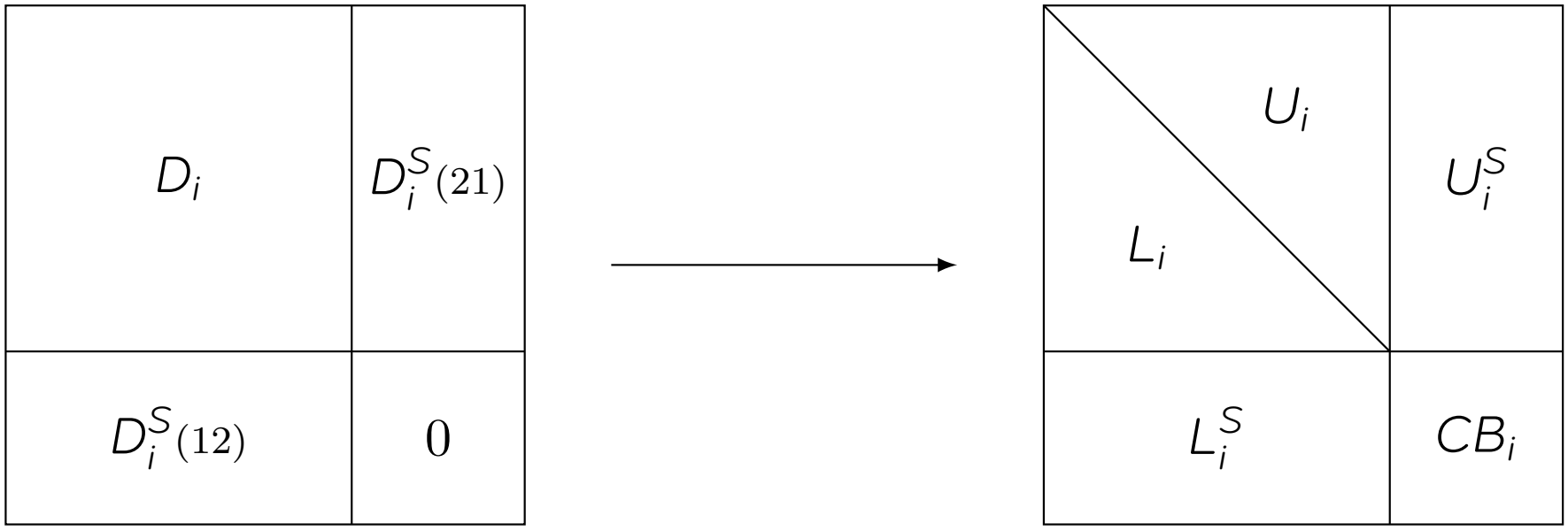
Nested dissection



D_1		
0	D_2	
D_1^S	D_2^S	S

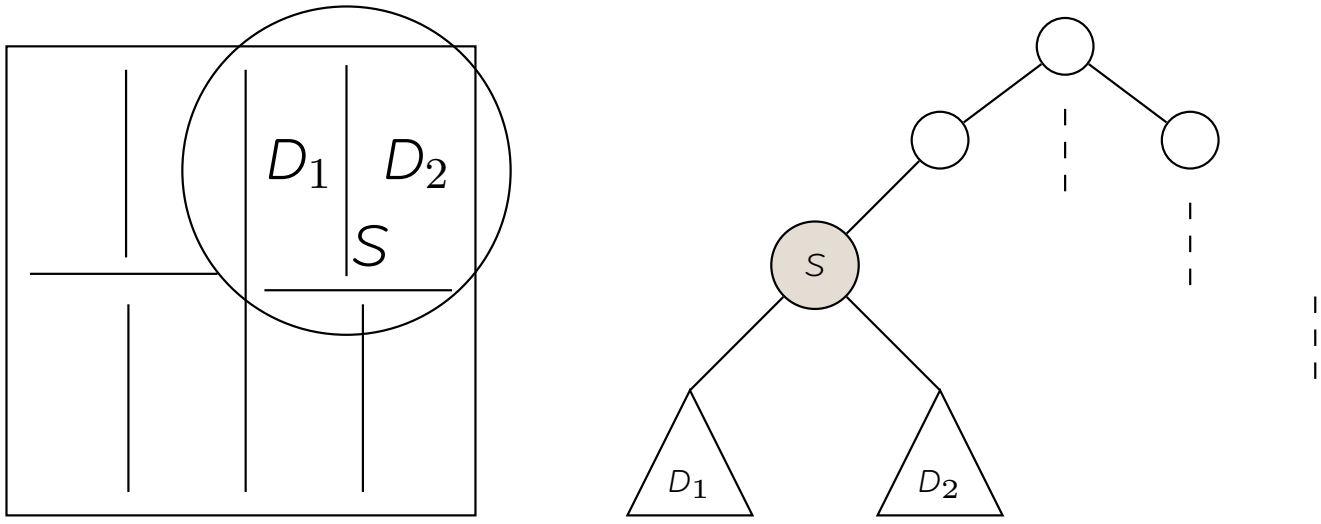
D_1 and D_2 are INDEPENDENT !

Elimination of D_i

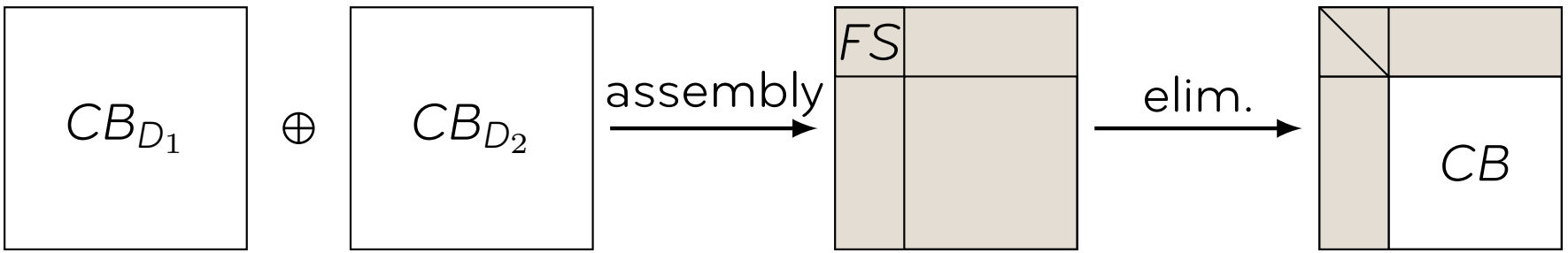


- $S \leftarrow S + CB_1 + CB_2$ *[assembly]*
- variables of S ready to be eliminated *[elimination]*

Generalization



At each node, a partial factorization of the frontal matrix is performed :



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Reveal and exploit low-rank structures within the fronts

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Why ?

- memory efficient : $A_{m,n} = U_{m,k} V_{n,k}^T + E_\varepsilon$ ("demote")
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The fronts are FULL rank

Solution :

Group variables to obtain low-rank *subblocks*

Grouping variables

Admissibility

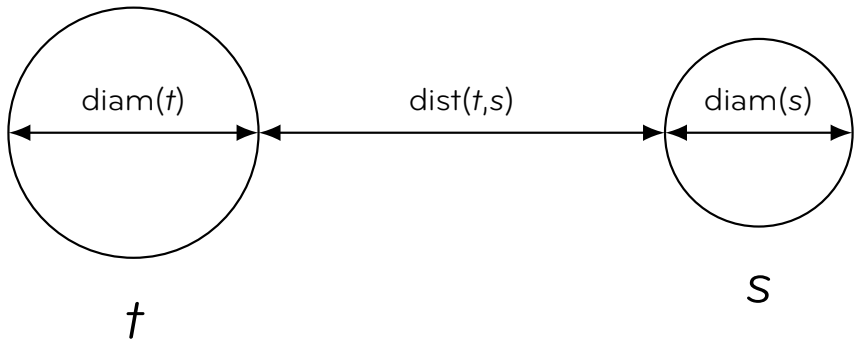
Is it *theoretically* worth demoting the block (t, s) ?

Admissibility condition for elliptic PDEs (Börm, Grasedyck, Hackbusch)

Let $t \subset \mathcal{I}$ and $s \subset \mathcal{J}$ be subsets of indices. The block (t, s) is said admissible if t and s satisfy

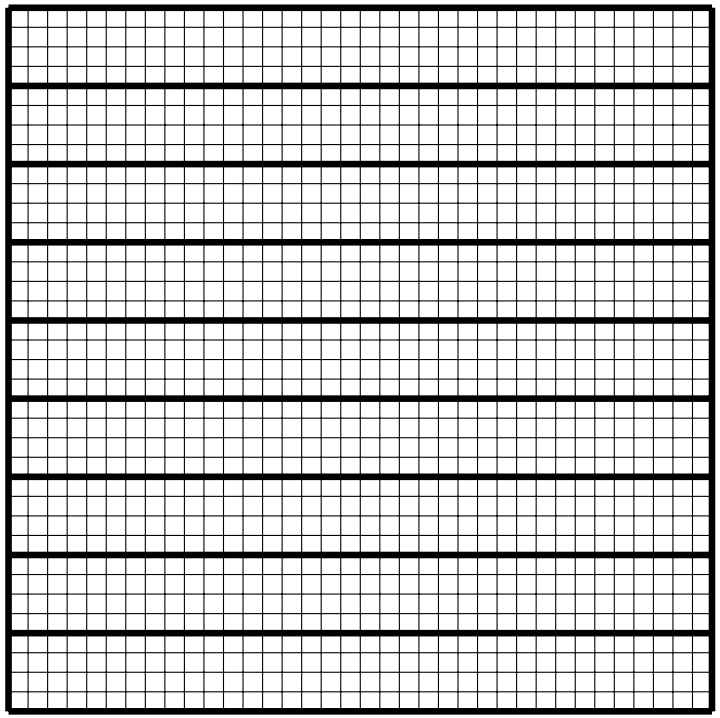
$$\text{diam}(t) + \text{diam}(s) \leq 2.\eta.\text{dist}(t, s)$$

where η is a problem dependant fixed parameter (usually 1 or 0.5)

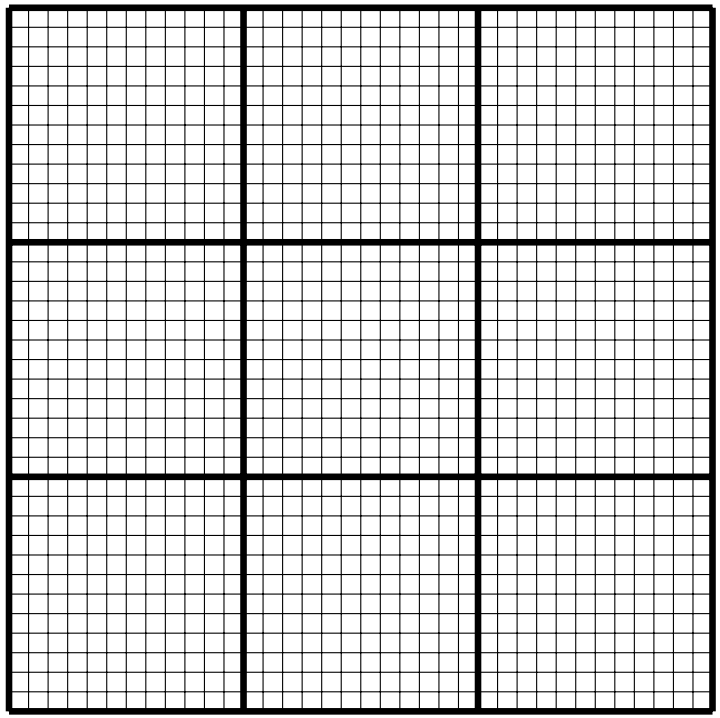


How to group the variables of a separator ?

Constraint : the admissibility condition should be satisfied

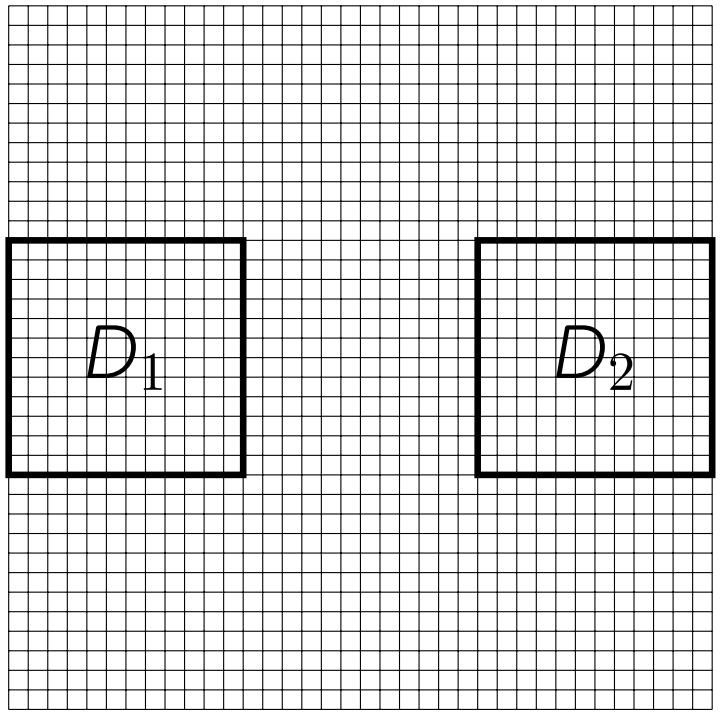


large diameters
fraction of memory used 83%



small diameters
fraction of memory used 57%

Rank vs distance (1)



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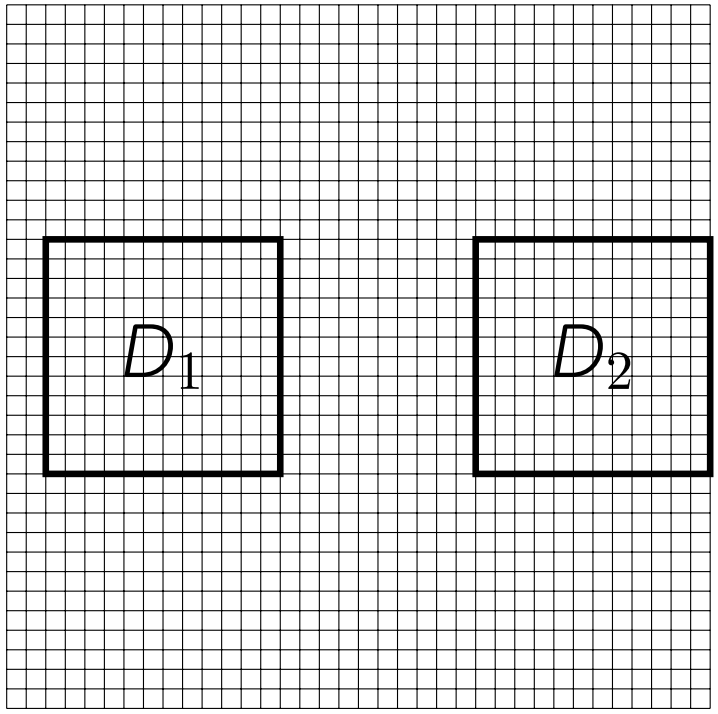

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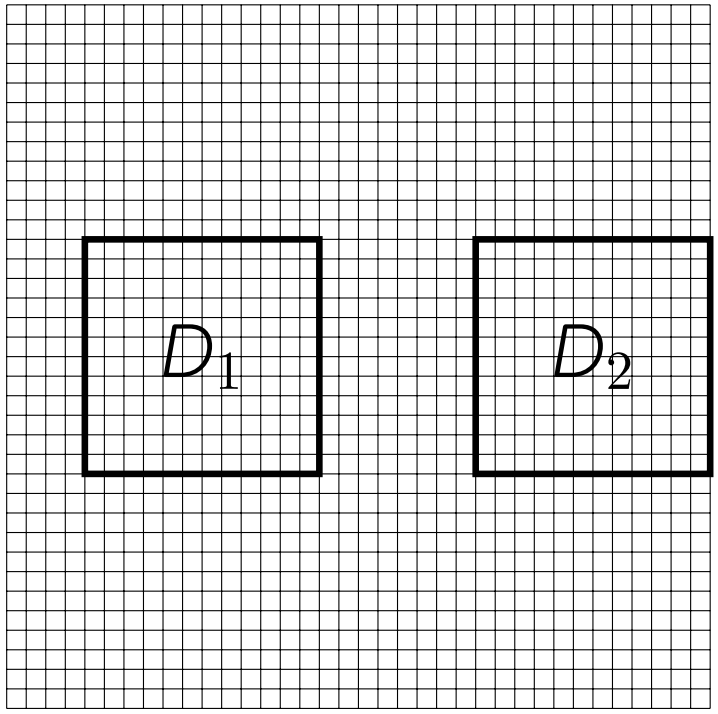

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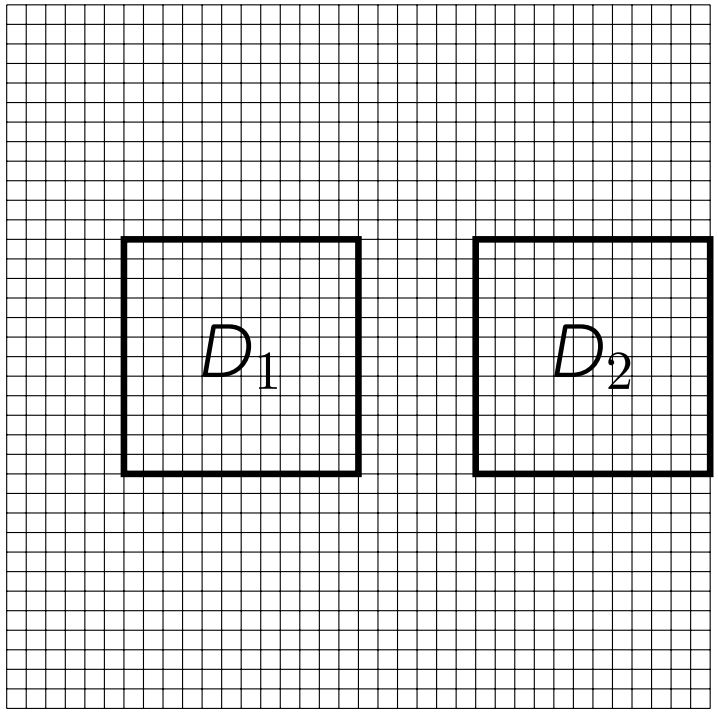

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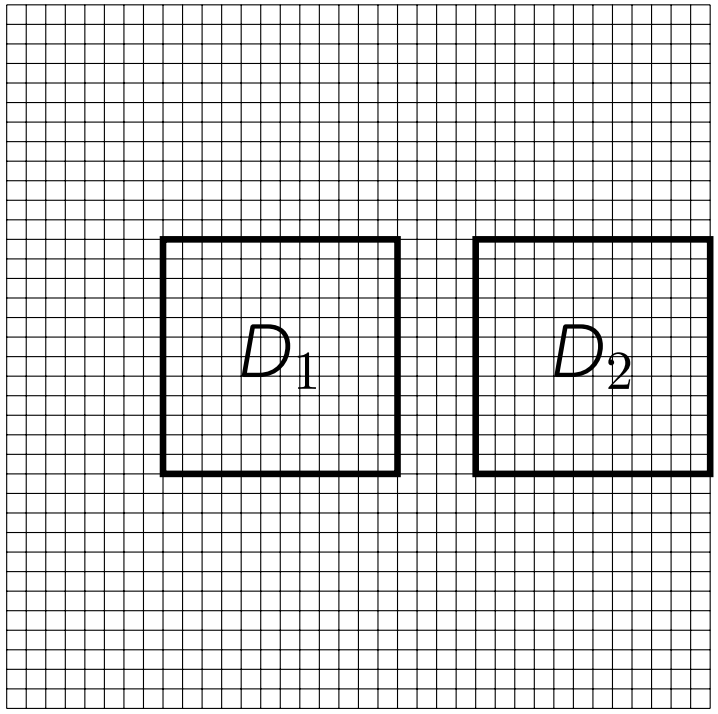

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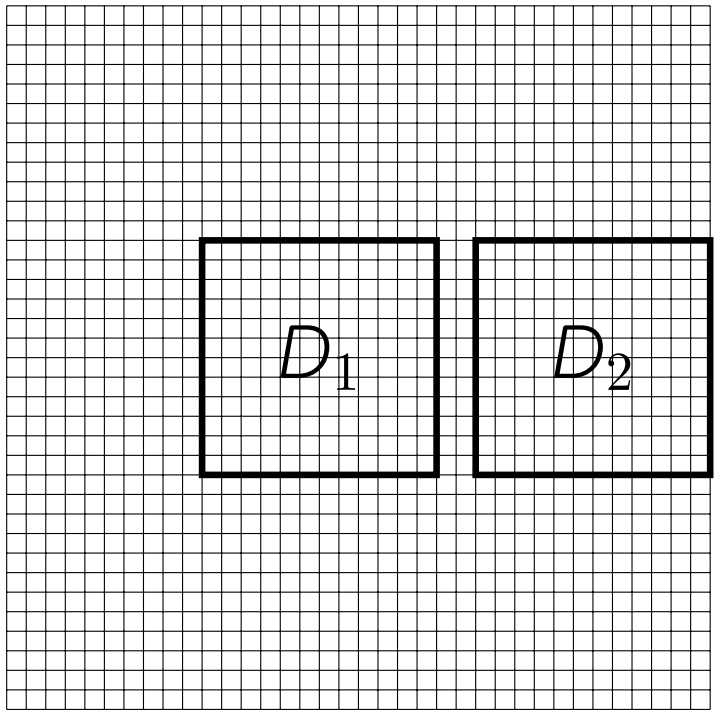
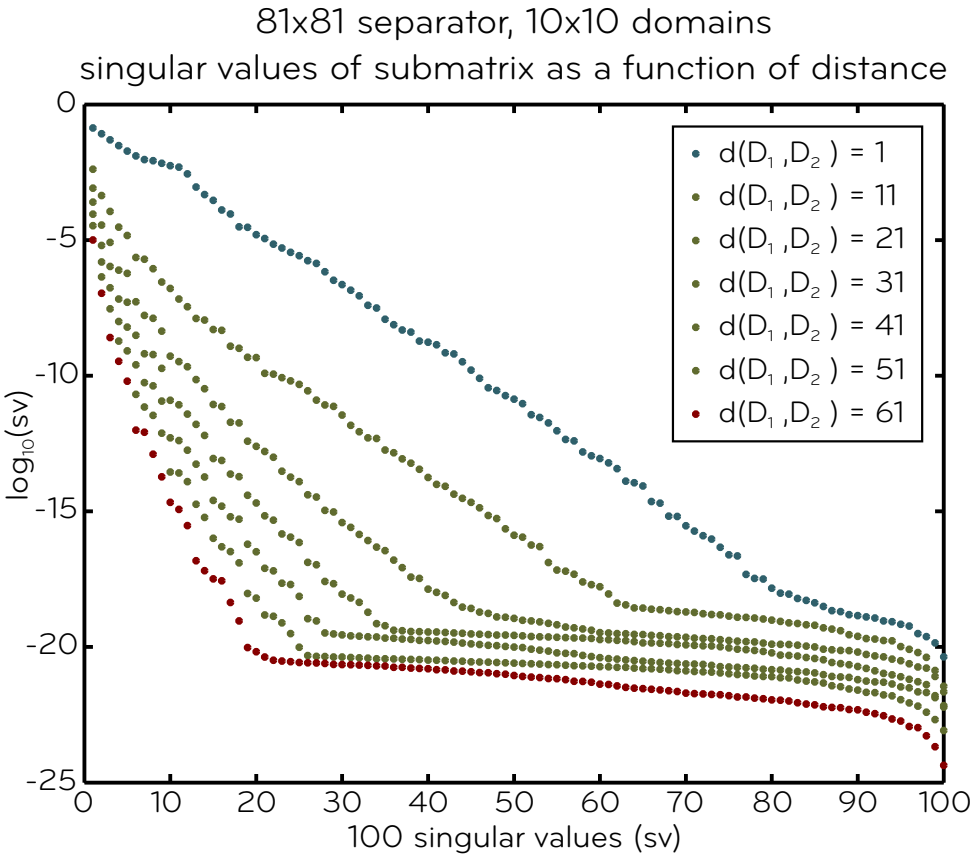
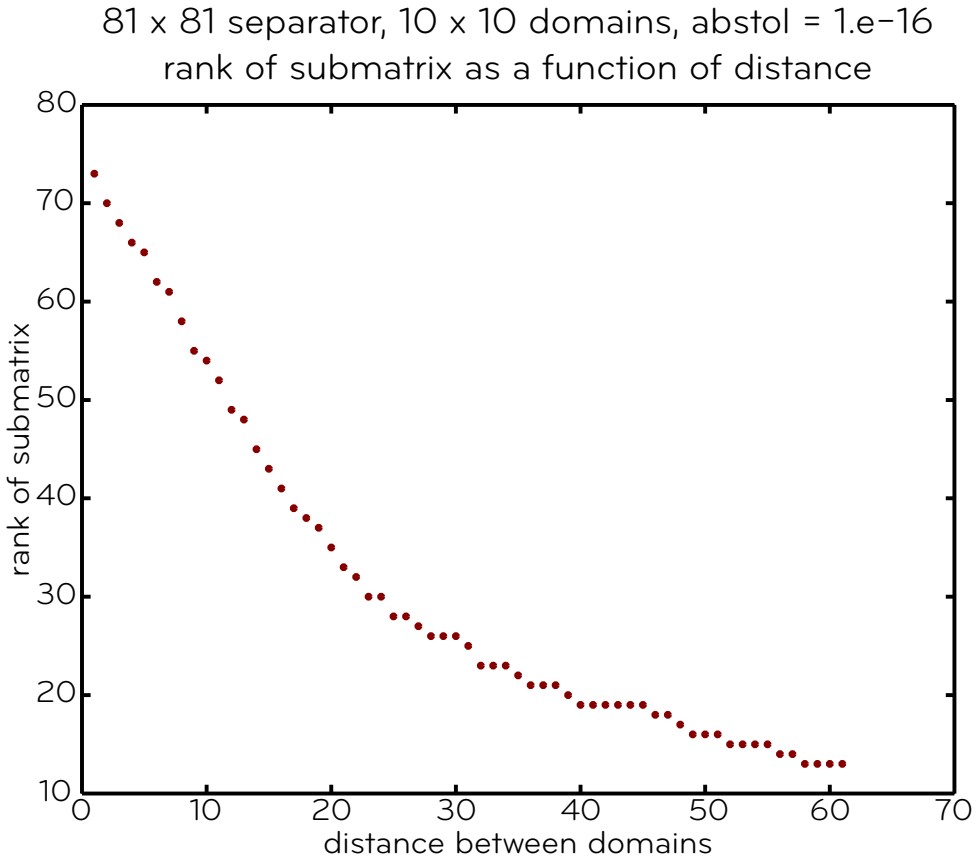


Rank vs distance (2)





Halo algorithm to group a separator

- Designed to catch the geometry of the problem
- Computed with the graph instead of the mesh
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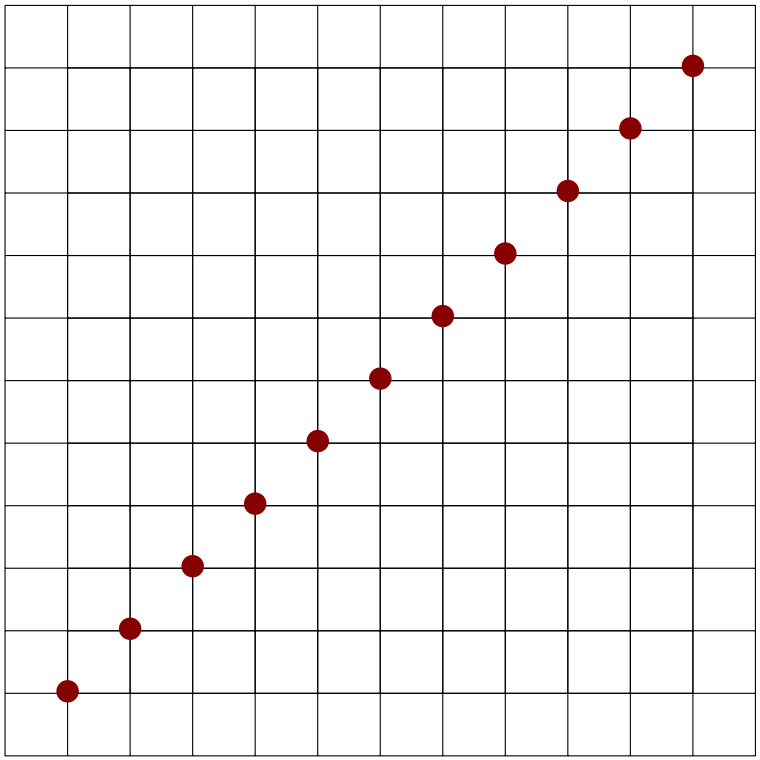

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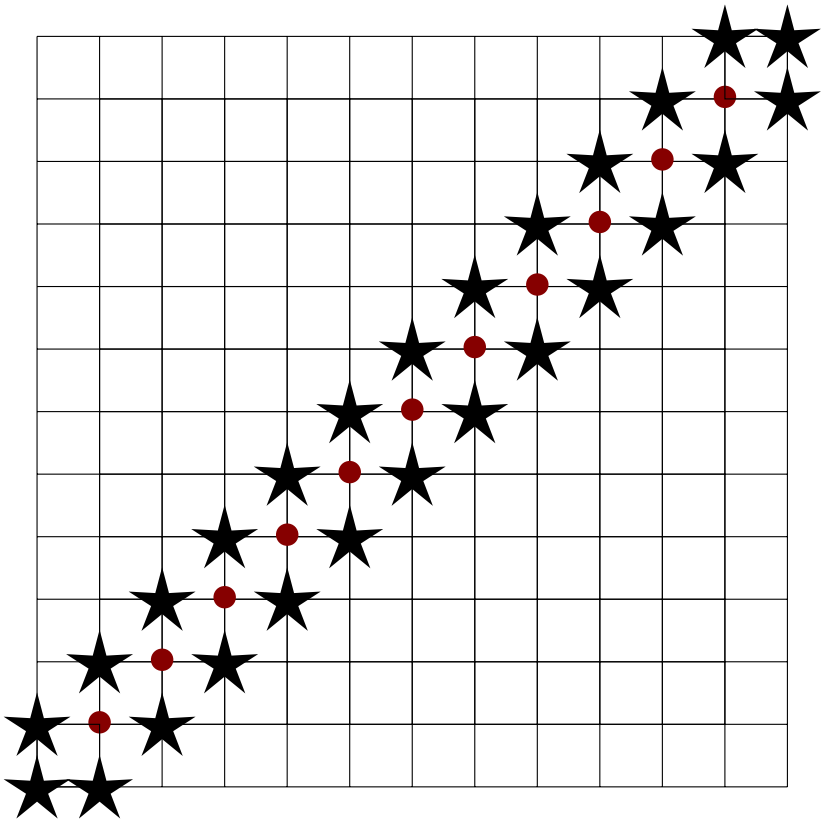

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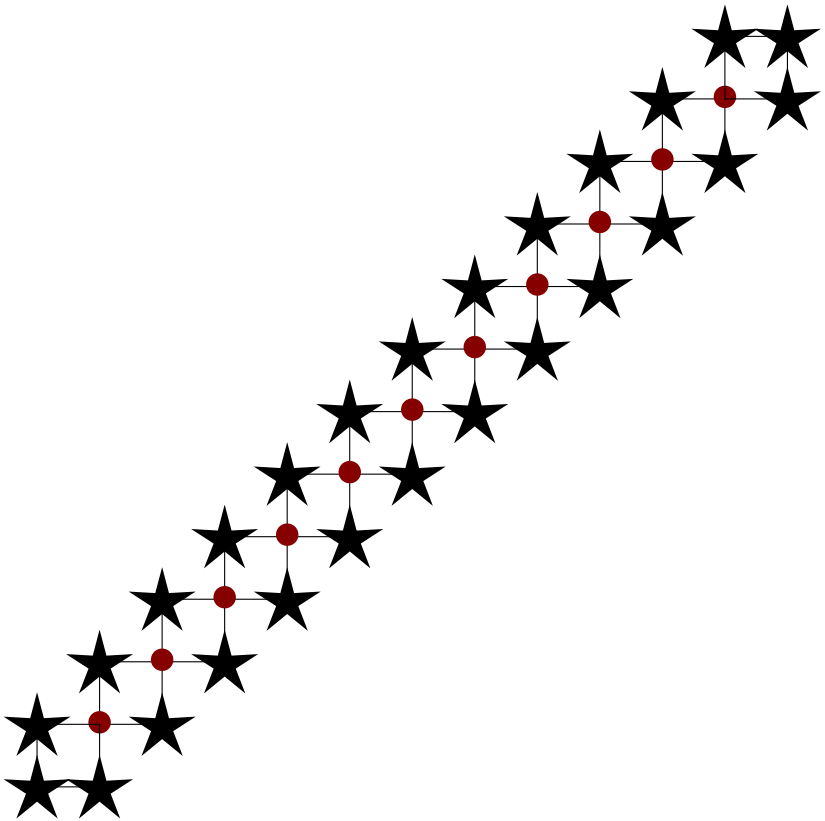

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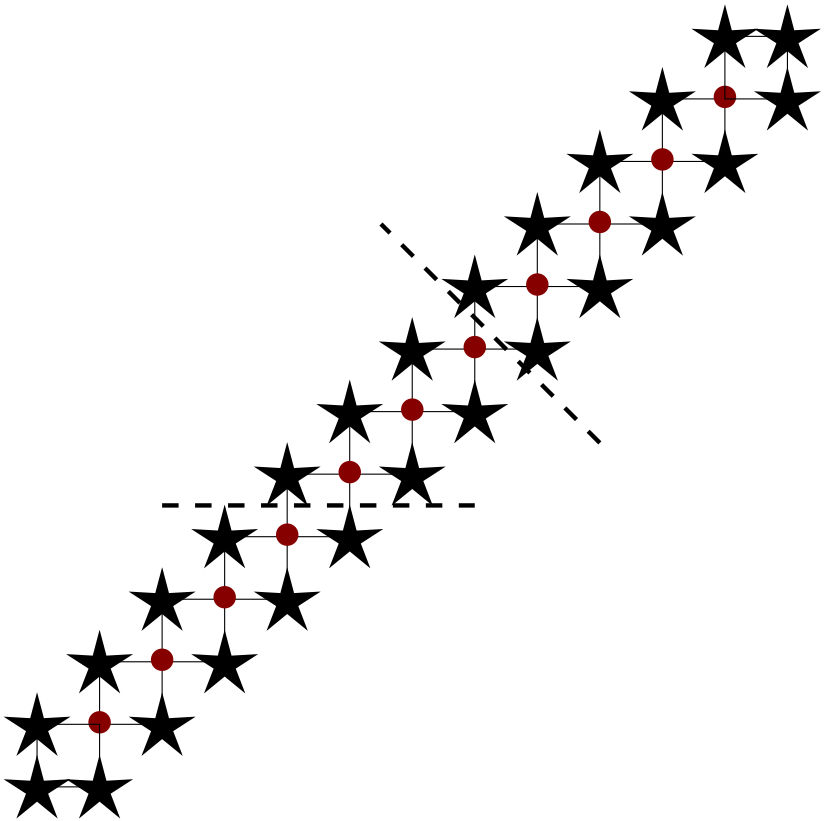

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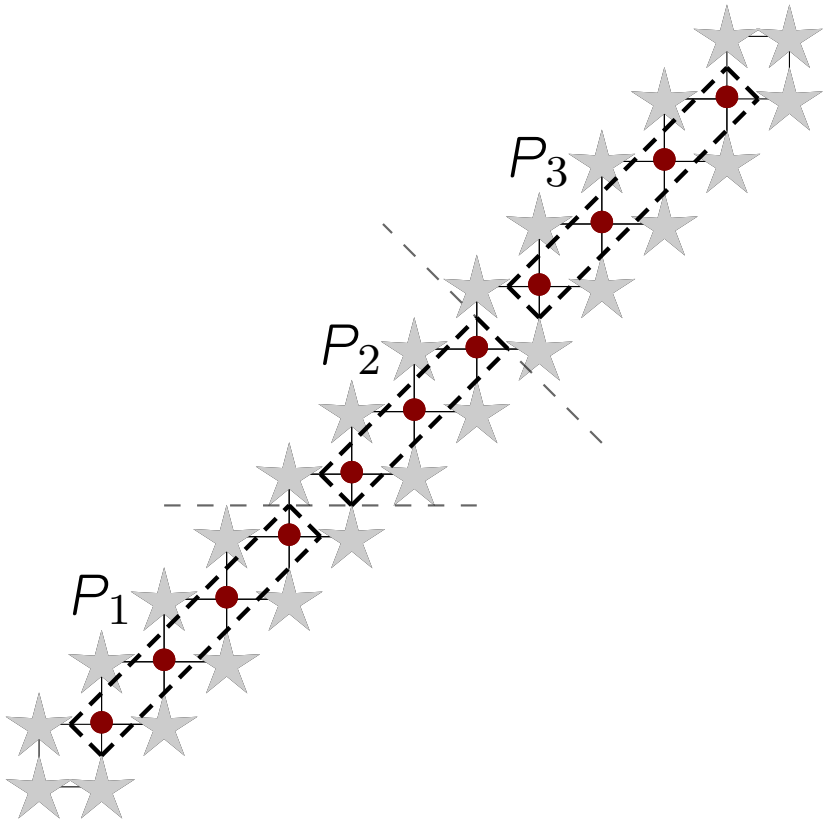
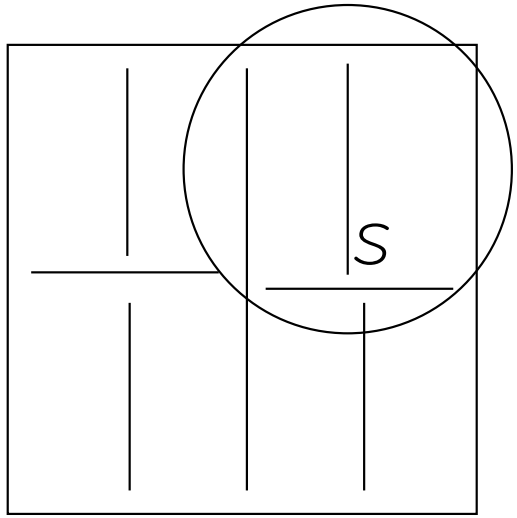


1. The separator
2. The halo
3. Extraction of the halo
4. Partition of the halo
5. Partition of the separator (block size is fixed)

How to group the variables of a front ?

$\Rightarrow$ front = separator + border



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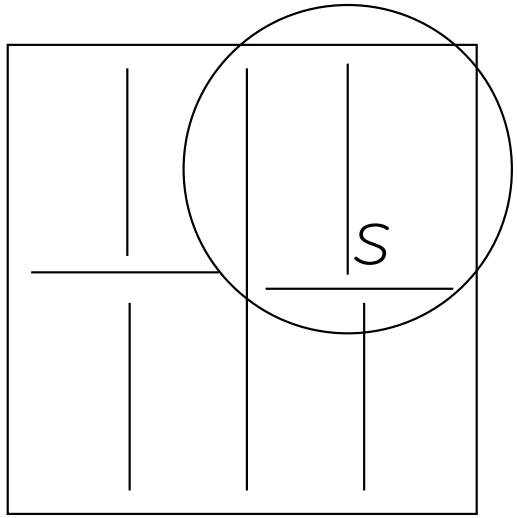

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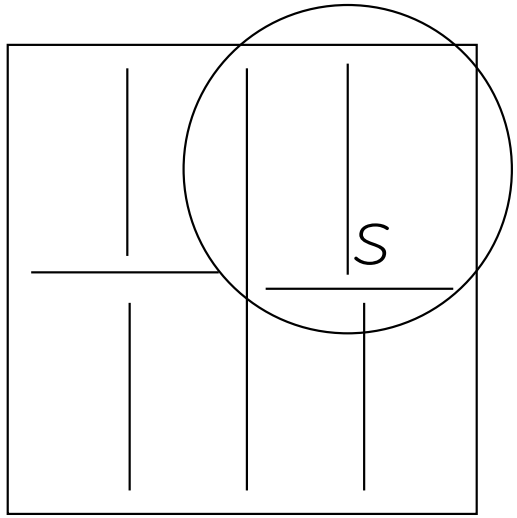

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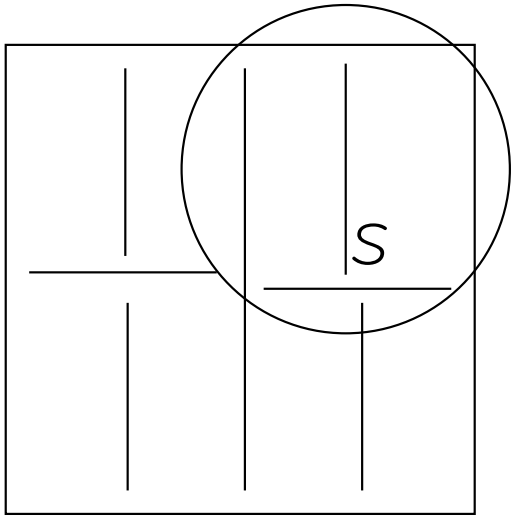

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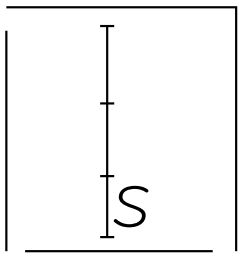

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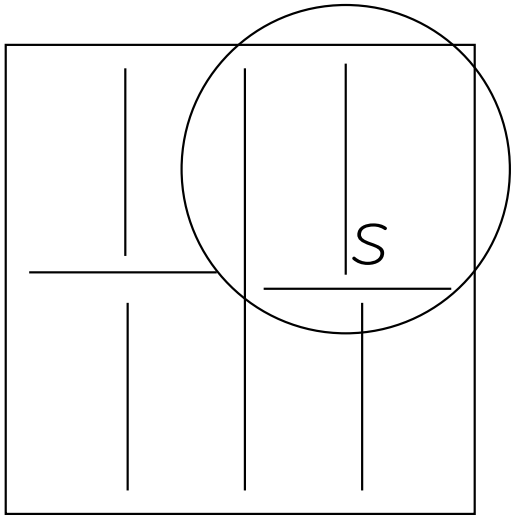

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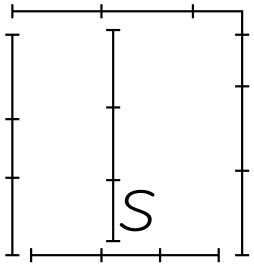

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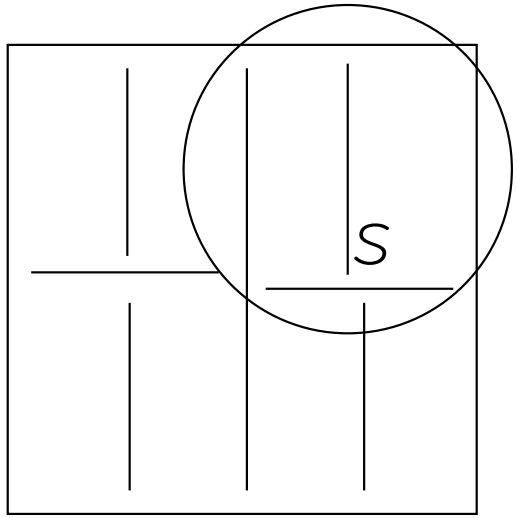

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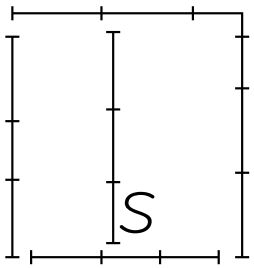

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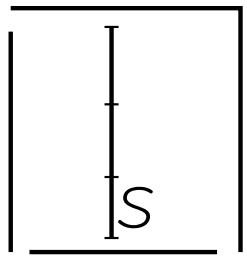

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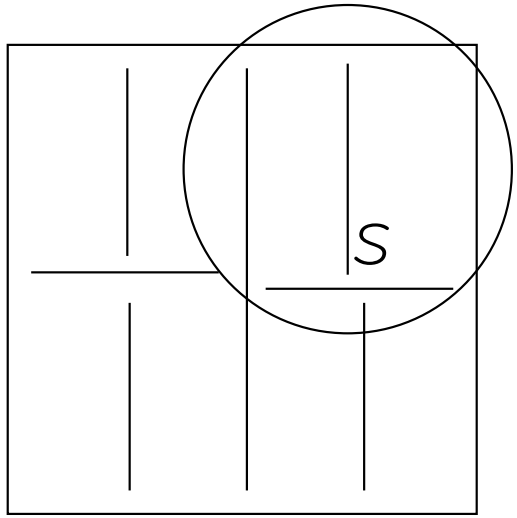

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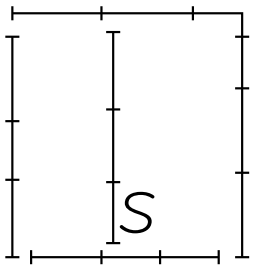

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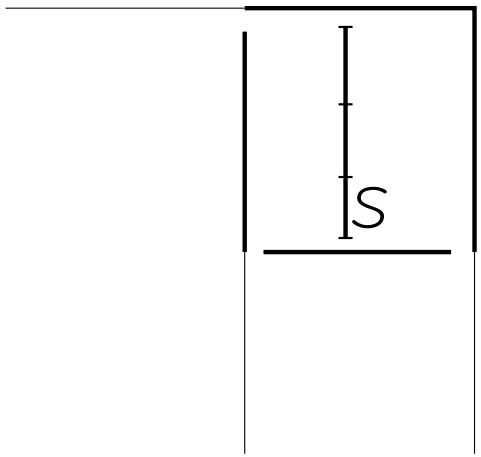

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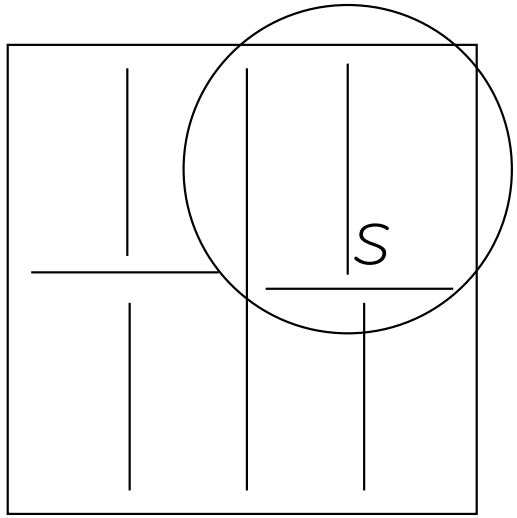

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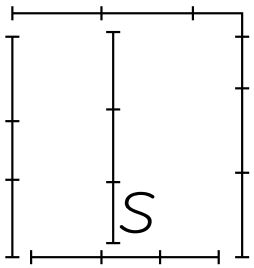

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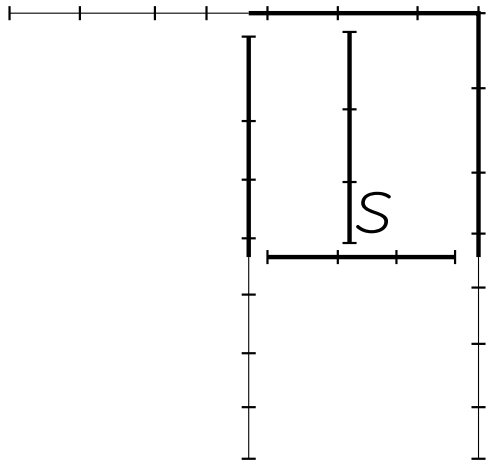

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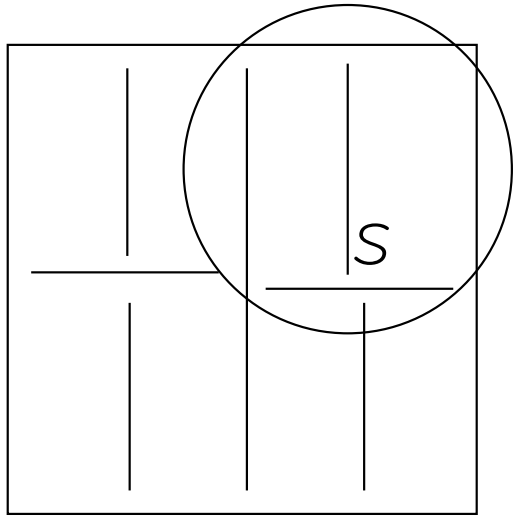

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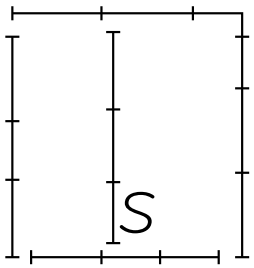

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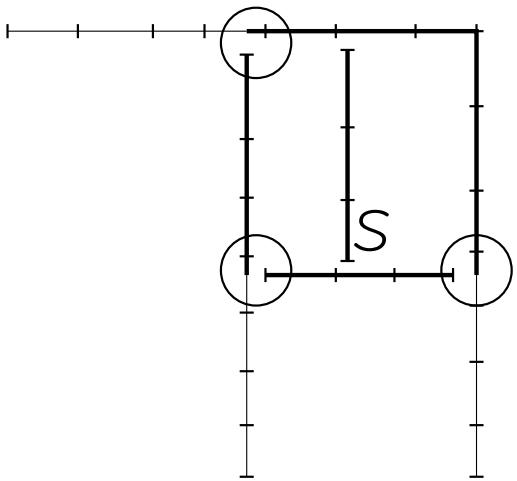

Comparison

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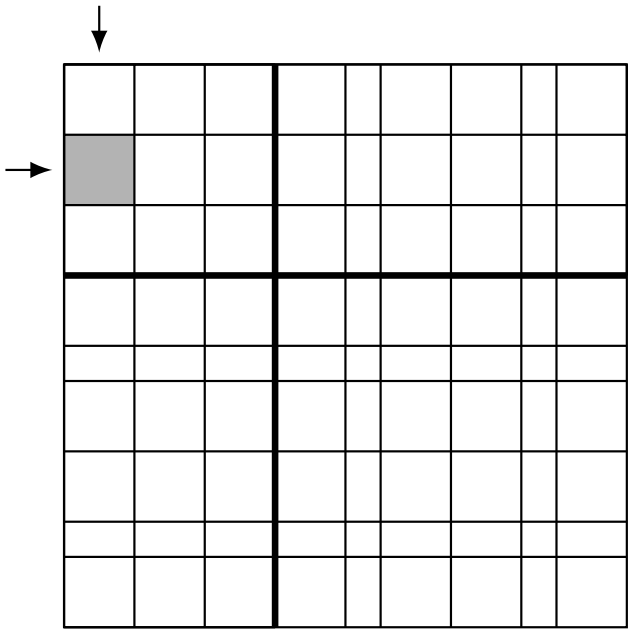
- same results on L_{11}



- optimal × optimal = optimal block

Comparison

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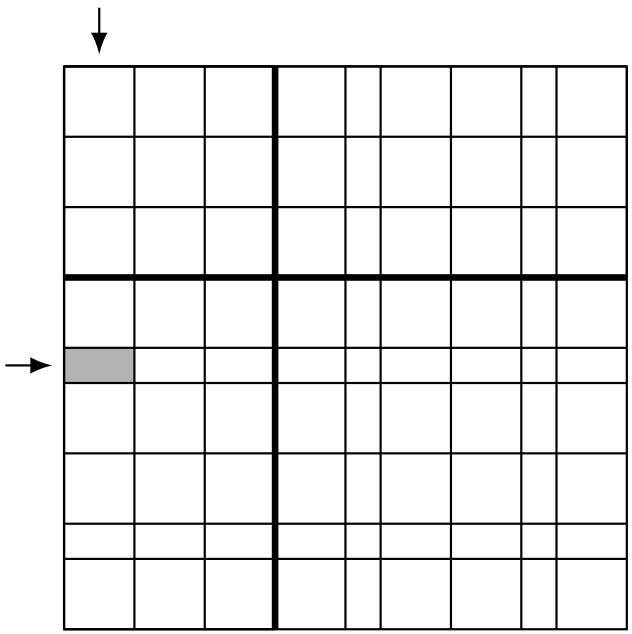
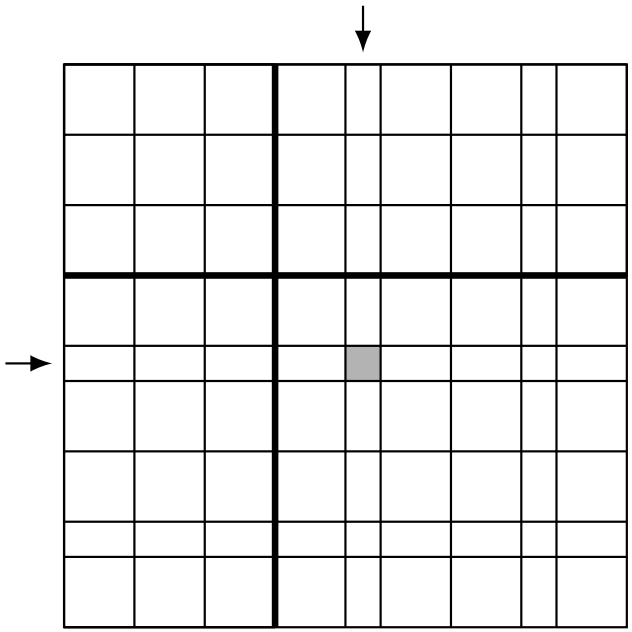
- same results on L_{11}
- comparable results on L_{21} ($\sim 2-3\%$ lost)



- optimal × optimal = optimal block
- small × optimal = large enough block

Comparison

- "inherited" version is more than 2 times faster
- same results on L_{11}
- comparable results on L_{21} ($\sim 2-3\%$ lost)
- a little less good on CBs ($\sim 10\%$ lost)

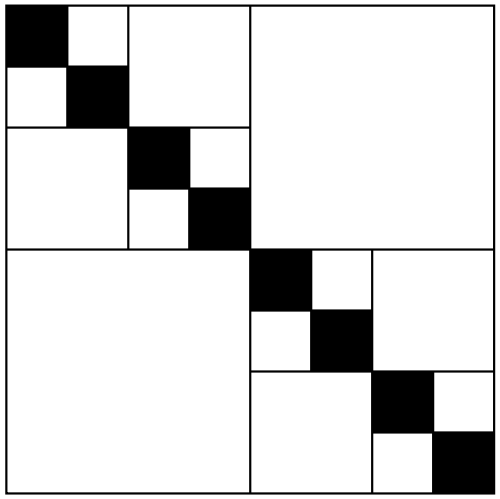


- optimal × optimal = optimal block
- small × optimal = large enough block
- small × small = too small block

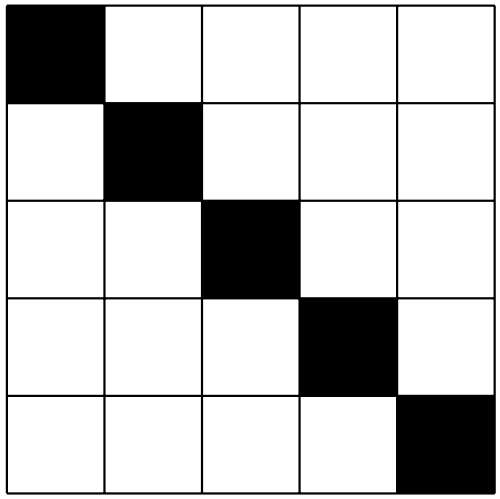
Exploiting low-rank blocks

Many matrix structures ...

$\mathcal{H}$ and $\mathcal{H}^2$ matrices (Hackbusch et al.), HSS matrices (Li, Napov, Xia et al.), HBS matrices (Gillman & Martinsson), BLR (us !) ...



HSS structure



BLR structure

Block Low-Rank (BLR) ...

- has less constraints for the grouping ;
- is more flexible (pivoting, ordering) ;
- seems more adapted to a multifrontal solver (small domains) ;

Experiments

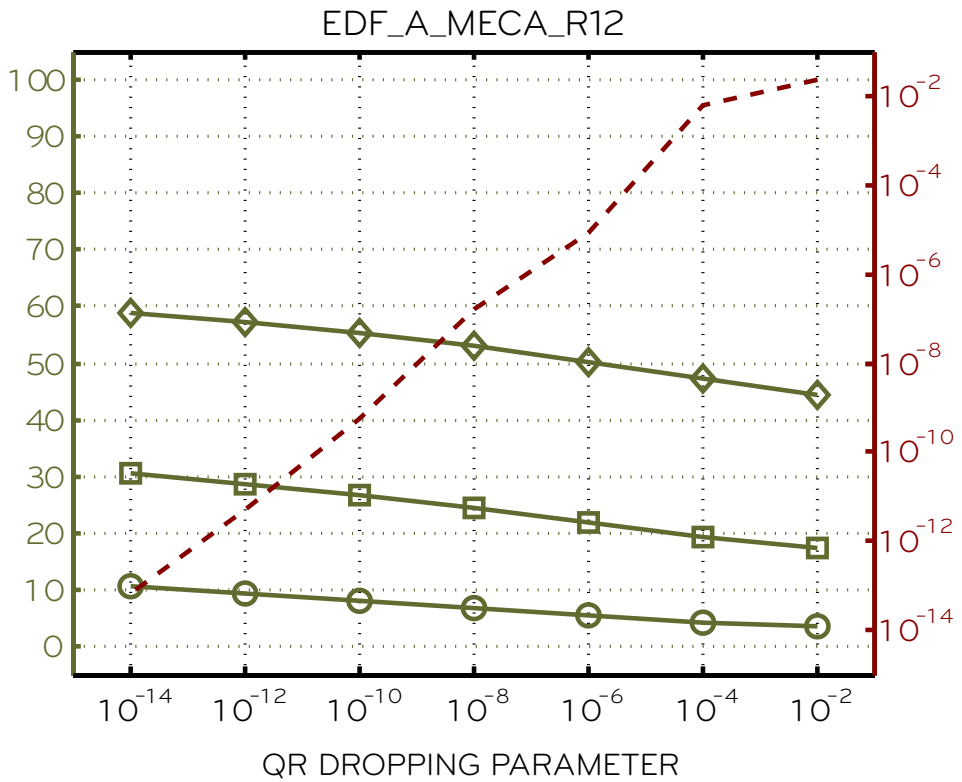
Set of problems

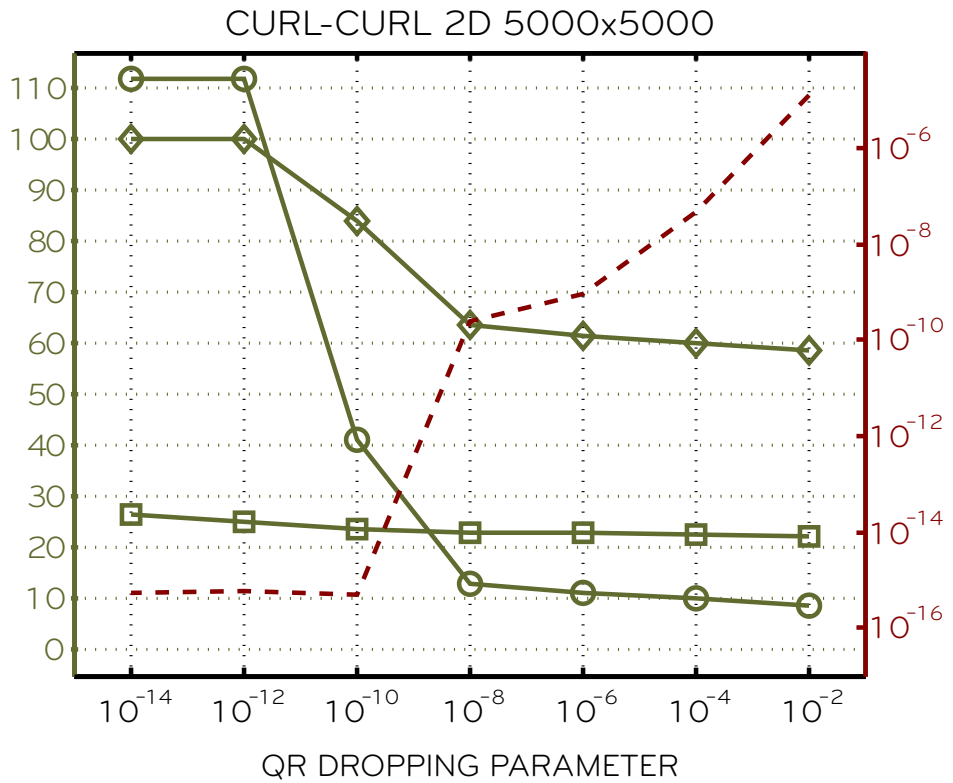
Name	Arith.	N	NZ	memory	flops	CSR$^{(1)}$	appli.
`Curl` 5000^2	D	50.10^6	2.10^8	29 GB	5.10^{12}	2.10^{-15}	∇
`Geoazur` 128^3	C	2.10^6	55.10^9	54 GB	6.10^{13}	3.10^{-4}	wave prop.
`EDF_A_MECA_R12`	D	134.10^6	1.10^9	151 GB	2.10^{14}	4.10^{-15}	mechanics
`EDF_D_THER_R7`	D	8.10^6	118.10^9	229 GB	1.10^{14}	8.10^{-15}	thermics

(1) CSR = Componentwise Scaled Residual = $\max_i \dfrac{|\mathbf{b} - \mathbf{A}\bar{\mathbf{x}}|_i}{(|\mathbf{b}| + |\mathbf{A}|\,|\bar{\mathbf{x}}|)_i}$

- 1 toy and 3 real industrial problems
- different applications
- target : harder and harder problems $\Rightarrow$ industrial partners

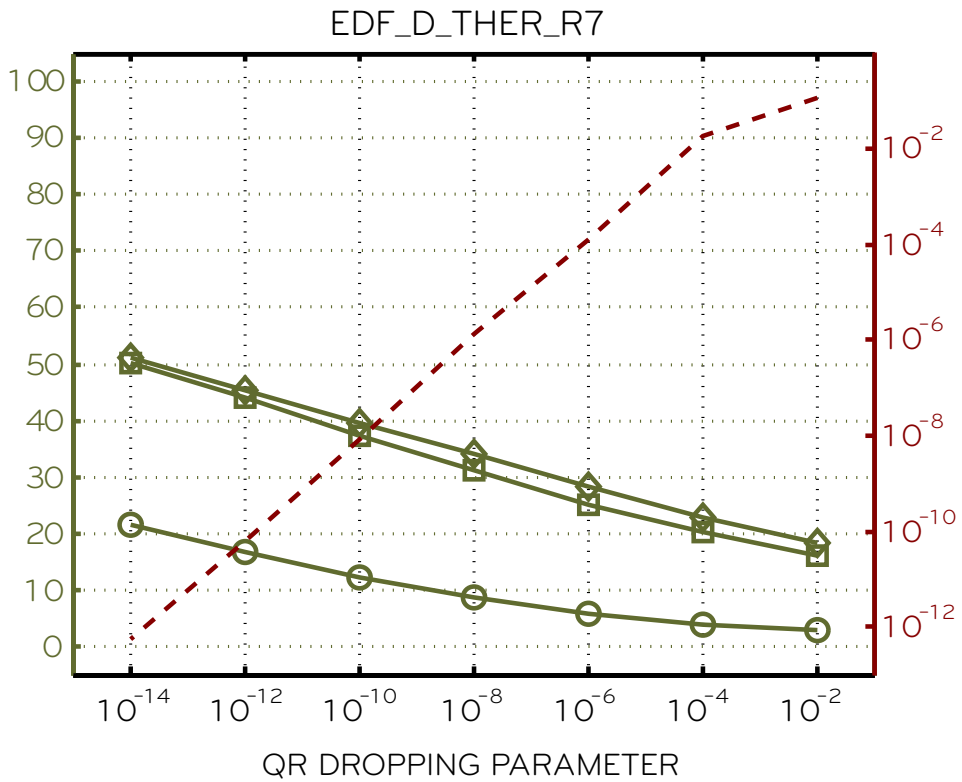
2D problems

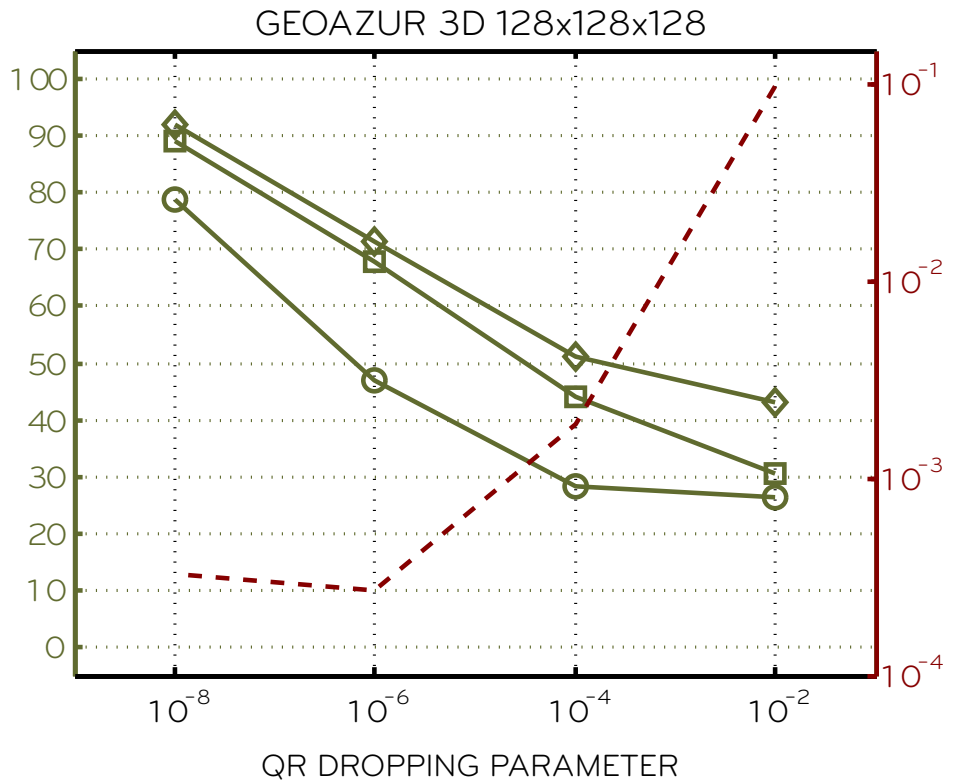




Name	N	memory	flops	CSR
EDF_A_MECA_R12	134.10^6	151 GB	2.10^{14}	4.10^{-15}
Curl-Curl 5000^2	50.10^6	29 GB	5.10^{12}	2.10^{-15}

3D problems





Name	N	memory	flops	CSR
EDF_D_THER_R7	8.10^6	229 GB	1.10^{14}	8.10^{-15}
Geoazur 128^3	2.10^6	54 GB	6.10^{13}	3.10^{-4}

Conclusion & perspectives

$\Rightarrow$ efficient method on applicative problems

- good memory reduction & large decrease in computations :
  150 GB $\rightarrow 60$ GB 7 H $\rightarrow 1$ H
- tree dependent method (any global ordering works)

$\Rightarrow$ can be used as a preconditioner or as an accurate solver

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- in parallel, significant part of memory consumption is temporary data

$\Rightarrow$ rooms for improvements (CB compression)

- pivoting
- diversify tested problems
- error propagation study

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¡Muchas gracias!

Any questions?